NOTE ON THE PICARD METHOD OF SUCCESSIVE APPROXIMATIONS.

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The Picard method of successive approximations, as applied to the proof of the existence of a solution of a differential equation of the first order, is commonly introduced somewhat after the following manner:

"We shall develop the method on an equation of the first order

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

supposing first that the variables are real. We shall assume that the function f is continuous when x varies from x_0 to $x_0 + a$ and when y varies between the limits $(y_0 - b, y_0 + b)$; that the absolute value of the function f remains less than a positive number M when the variables x, y remain within the preceding limits; and, finally, that there exists a positive number A such that we have

$$|f(x, y) - f(x, y')| < A|y - y'|$$

for any positions of the points (x, y) and (x, y') in the preceding region.

"Let us suppose, for ease in the reasoning, $a > 0$, and let h be the smaller of the two positive numbers a, b/M. We shall prove that *the equation* (1) *has an integral which is continuous in the interval* $(x_0, x_0 + h)$ *and which takes on the value* y_0 *for* $x = x_0$."

This particular language is quoted substantially from Goursat's Mathematical Analysis, translated by Hedrick and Dunkel,* except that the statement there is for a pair of differential equations in two unknown functions. The italics are kept from the original French. After the proof has been given, the following remark is added:†

"If $\cdots$ we go over the proof again, we see that the condition $h < b/M$ is needed only to make sure that the intermediate functions y_1, y_2, $\cdots$ [the successive approximations to the solution] do not get out of the interval $(y_0 - b, y_0 + b)$, so that the functions $f(x, y_i)$ shall be continuous functions of x between x_0 and $x_0 + h$. If the function $f(x, y)$ remains continuous when x varies from x_0 to $x_0 + a$, and when y varies from $-\infty$ to $+\infty$, it is unnecessary to make this requirement."

* Vol. 2, part 2, pp. 61–62.

† Loc. cit., p. 64; the statement is again simplified from two unknowns to one, in quoting.

The purpose of this note is to point out that even if $f(x, y)$ is originally defined only in a rectangle, it is a simple matter to extend its definition outside the rectangle, so that the conditions of the hypothesis shall hold for $x_0 \leqq x \leqq x_0 + a$ and for all real values of y, the Lipschitz condition as well as the mere continuity. It is sufficient, for example, to let

$$\begin{aligned} f(x, y) &= f(x, y_0 + b), \qquad & y &\geqq y_0 + b, \\ f(x, y) &= f(x, y_0 - b), \qquad & y &\leqq y_0 - b. \end{aligned}$$

The process of successive approximations then gives, at a single stroke, a function $y(x)$ which is defined and satisfies the differential equation, with the extended definition of $f(x, y)$, for $x_0 \leqq x \leqq x_0 + a$. It satisfies the *original* equation as long as the x and y *of the solution* remain within the original rectangle, whatever the behavior of the approximating functions may be. The solution is unique as long as it stays in the rectangle. The original equation of course has no authority outside its own domain, and corresponding to the infinitely many possible ways of extending the definition of f there will be infinitely many different extensions of the solution, if it leaves the rectangle before x reaches $x_0 + a$.

It may seem that this observation is trivial, and it is perhaps hardly probable that it is made here for the first time;* but its omission from standard presentations of the subject is notable. In the treatise already quoted, for example, after the Cauchy-Lipschitz proof has been explained, the two demonstrations are compared as follows:†

"Cauchy's first method [the Cauchy-Lipschitz method] and that of the successive approximations give, as we see, the same limit for the interval in which the integral surely exists. But from a theoretical point of view Cauchy's method is unquestionably superior: we shall show, in fact, that this method enables us to find the integral in every finite interval in which the integral is continuous."

Again, the Encyclopédie des sciences mathématiques, in the article *Existence de l'intégrale générale. Détermination d'une intégrale particulière par ses valeurs initiales*, after going into some detail on the question of the length of the interval of convergence, says:‡

"On ne connaît encore aucun moyen de déterminer l'intervalle exact dans lequel la méthode de E. Picard converge. Suivant les cas, cet intervalle peut embrasser, comme dans la méthode de Cauchy-Lipschitz, tout l'intervalle de régularité de la solution, ou être au contraire plus petit que l'intervalle de convergence des séries de Taylor en $(x - x_0)$ qui représentent la solution, quand elle est holomorphe."

* Since this note was written, I have learned that Professor Wedderburn made essentially the same suggestion, in unpublished form, a number of years ago.

† Loc. cit., p. 73.

‡ Tome 2, volume 3, fascicule 1, pp. 14–15.

While the present remarks do not perhaps invalidate either of these statements, it does seem fair to say that they have a bearing on the comparison.

It is readily seen that the method outlined above can be extended to the case of a system of n differential equations in n unknown functions. If there are two equations, for example,

$$\frac{dy}{dx} = f(x, y, z), \qquad \frac{dz}{dx} = \varphi(x, y, z),$$

with right-hand members defined for $x_0 \leqq x \leqq x_0 + a$, $y_0 - b \leqq y \leqq y_0 + b$, $z_0 - c \leqq z \leqq z_0 + c$, it is possible to set

$$\begin{aligned}
f(x, y, z) &= f(x, y_0 + b, z), && y \geqq y_0 + b, \quad z_0 - c \leqq z \leqq z_0 + c; \\
f(x, y, z) &= f(x, y, z_0 + c), && y_0 - b \leqq y \leqq y_0 + b, \quad z \geqq z_0 + c; \\
f(x, y, z) &= f(x, y_0 + b, z_0 + c), && y \geqq y_0 + b, \quad z \geqq z_0 + c;
\end{aligned}$$

and similarly in the other regions of the yz-plane, with a corresponding treatment for φ. More concisely, for any number of dimensions, the value of each function at any point outside its original domain of definition is to be the same as the value which it has at the nearest point of that domain.

The same method, though of course not the same formulas, can be used even if the original domain is not rectangular, provided that it has a moderately regular boundary, so that the functions can be extended across the boundary with the requisite degree of continuity.

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